GENERATING EXAMPLES: FROM PEDAGOGICAL TOOL TO A RESEARCH TOOL

RINA ZAZKIS, ROZA LEIKIN

We all learn from examples. By seeing a picture of giraffes and recognizing their invariant properties, a child constructs a general concept of a giraffe and, in such, is able to distinguish a giraffe from, for example, a unicorn. Further, by looking at a giraffe drawn by a child, we notice what critical features of this creature have been identified and what may have been omitted.

We acknowledge two important roles that examples play in mathematics education. One is of interest to teachers and designers of instructional materials, while the other is of interest to researchers. Whereas the pedagogical aspect has been broadly discussed (*e.g.*, Watson and Mason, 2005; Zhu and Simon, 1987; Leinhardt, 1993), our goal, in this article, is to introduce the power of examples as a research tool that provides a 'window' into a learner's mind.

According to Skemp (1987), a concept, for its formation, requires a number of experiences – or examples – that have something in common. Examples are considered to be an integral part of teachers' instructional explanations (Leinhardt, 1993). Thus, both teachers and their students have experienced sets of examples corresponding to what they teach and learn.

In considering *learning from examples* in the context of mathematics education, the most common reference is to "worked examples" (*e.g.*, Zhu and Simon, 1987), that is, explicit solutions to exercises shown by an instructor or provided in a text. These examples are supposed to demonstrate the use of specific techniques, which are in turn to be mimicked or lightly modified by students in dealing with similar exercises. In this case, examples are provided by an authority (teacher or textbook) and those who learn from examples are students.

We highlight a different role of examples, by switching the positions between those who provide examples and those who learn from them. In our case, researchers are those who learn from examples, whereas examples are generated by research participants, who can be either students, student-teachers or practising teachers. We suggest that from the participants' repertoire of examples researchers may learn about their knowledge, both mathematical and pedagogical.

We believe that examples generated by participants – if solicited in a certain way – mirror their conceptions of mathematical objects involved in an example generation task, their pedagogical repertoire, their difficulties and possible inadequacies in their perceptions. However, there is a need for explicit criteria for evaluating examples generated by participants. We suggest such criteria that provide a lens through which participant/learner generated examples can be examined for the purpose of researchers' learning.

Examples and example spaces

We build on the recent work of Watson and Mason (2005), which provides a comprehensive treatment of the examples used in teaching and learning mathematics. Watson and Mason focus on learner generated examples, a teaching strategy of asking learners to construct their own examples of mathematical objects under given constraints. They claim, and provide ample evidence, that learner generated examples serve as a powerful pedagogical tool for enhancing the learning of mathematics at a variety of levels.

Further, Watson and Mason introduce the idea of example spaces, which are collections of examples that fulfill a specific function. They suggest that example spaces are influenced by individual's experience and memory, as well as by the specific requirements of an example generating task. They explain that their "particular interest is in how learners' example spaces emerge and develop as they look for particular examples in response to prompts" (p. 59). They use a metaphor of a larder to explain ideas related to example generation: like objects in a larder, some examples are immediately accessible, others require some effort to be reached and some may be out of reach for a particular person or in a particular task.

Watson and Mason (p. 76) distinguish between several kinds of example spaces:

- *situated (local), personal (individual) example spaces*, triggered by a task, cues and environment as well as by recent experience
- *personal potential example space*, from which a local space is drawn, consisting of a person's past experience (even though not explicitly remembered or recalled), which may not be structured in ways that afford easy access
- *conventional example space*, as generally understood by mathematicians and as displayed in textbooks, into which the teacher hopes to induct his or her students
- *a collective and situated example space*, local to a classroom or other group at a particular time, that acts as a *local conventional space*.

We elaborate upon these distinctions in developing a framework to allow the making of inferences about one's knowledge by analyzing participants' example spaces and comparing the personal and the conventional example spaces. However, to provide a context for such a framework, we first consider several examples of example generation.

Odd numbers

The following excerpt presents a student's (all participants' names are pseudonyms) attempts to give an example of a number that leaves a remainder of 1 in division by 2. The interviewee is a pre-service elementary school teacher. The interview was conducted as part of ongoing research into pre-service elementary teachers' understanding of elementary number theory. The task presented to the participant was not designed to investigate her understanding of division with remainders or even and odd numbers. It was intended to be a 'warm-up' task that was supposed to put the interviewee at ease by exploring familiar territory (Ginsburg, 1997).

1. Interviewer: Can you please think of a 5-digit number that leaves a remainder 1, when divided by 2?
2. Cindy: [Pause] I'm thinking it would probably have to be an odd number, because all even numbers would be evenly divisible by 2 . . .
3. Interviewer: OK . . .
4. Cindy: And, [(pause], I'm trying to think of what number to put on the ends, but I'll have 1 [pause], I don't, actually maybe it's not possible, I don't know . . .
5. Interviewer: What is not possible?
6. Cindy: To have a remainder of 1, but . . .
7. Interviewer: You said a moment ago something about even and odd . . .
8. Cindy: It couldn't be an even number . . .
9. Interviewer: It cannot be an even number, so it must be an odd number . . .
10. Cindy: Um hm . . .
11. Interviewer: So when you know that it must be an odd number, what do you think about now?
12. Cindy: Well I think of the prime, actually not prime, but, [pause] I don't know, I'm probably stumped. Uh, [pause] I guess maybe just look at simpler cases, just look at 3 and 5 and 7 and . . .
13. Interviewer: 3, 5 and 7, OK, there are simpler cases when you look at them . . .
14. Cindy: [pause] 2 is in the 3 once, remainder 1 . . .
15. Interviewer: [pause] Okay, so you have written the number which is 10,003. You divided by 2, and this is your answer: 5001, remainder 1. Oh, it was hard, was it?
16. Cindy: [laugh] [pause]
17. Interviewer: Can you give me another number with 5 digits, that when divided by 2 has a remainder 1?
18. Cindy: I'll have to play around with those numbers. I'd keep 3 on the end . . .

In contribution 2, Cindy demonstrates a clear connection between the remainder of 1 in division by 2 and odd numbers. This connection is established by elimination, that is, "even numbers would be evenly divisible by 2", and, as such, it is implied that even numbers leave no remainder, reducing the possible examples to odd numbers. This view is also repeated in contribution 8. In contribution 4, Cindy refers to the last digit of a number, mentioning "what number to put on the ends". She is likely to be distinguishing between even and odd numbers based on their last digit. In contribution 12, Cindy mentions "prime", probably having a momentary confusion between prime and odd. Being "stumped", Cindy employs a powerful problem solving strategy – consideration of simpler but similar cases – attending to numbers 3, 5 and 7. In contribution 14, she explicitly verifies that her example of 3 satisfies the requirement of a number having a remainder of 1 in division by 2. Having verified for 3, Cindy checks the number 10,003. In contribution 15, the interviewer explains what Cindy has done and after a short pause asks for another example in contribution 17. Cindy's reply (contribution 18), "I'll have to play around", is a clear indication of her intention to generate such examples by trial and error. Having faced success with using 3 as the last digit, she intends to keep this strategy.

In this excerpt, the limited pool of examples and lack of fluency in the way they are generated provides a reasonable illustration of Cindy's conceptual structure: she understands the implication, if a number leaves a remainder of 1 in division by 2 then it is an odd number. However, the inverse implication, every odd number satisfies the requested condition, is missing. Consequently, she experiences difficulty in exemplifying the general observation with a specific example.

We are not claiming that Cindy's approach exemplifies a popular misconception. Our only claim is that a specific example generation task serves as a tool for researchers to describe Cindy's understanding of the underlying concepts. The issue of accessibility and correctness of specific examples is acknowledged in the framework we propose.

Arithmetic sequence

A study of Zazkis and Liljedahl (2002) focused on pre-service teachers' understanding of the additive and multiplicative structure of arithmetic sequences. Starting with a straightforward and comfortable task, participants were asked at the beginning of a clinical interview to provide several examples of arithmetic sequences. With the exception of the sequence of odd numbers, all the examples generated by the participants were sequences of multiples with a constant difference (*e.g.*, 3, 6, 9, 12, … or 10, 20, 30, …). A request for 'something different' resulted either in increasing the numbers for the constant difference or in an example for a decreasing sequence. Although all the participants recognized the sequence 2, 5, 8, …, as arithmetic, sequences of so called 'non-multiples' were not in their personal evoked example space. Consequently, an example-generation task, chosen as an easy 'warm up' for an interview, provided a fruitful direction for the analysis of the data. In the analysis, major differences were found in participants' responses related to sequences of multiples versus sequences of non-multiples, with a greater success and more confidence displayed in tasks related to the former.

Irrational numbers

In this study, we look at several sets of examples of irrational numbers. In the last case, there was a specific request to give an example of an irrational number between 100 and 200. In other cases, the interval from which examples were to be drawn was not specified.

The following excerpt is from the interview with Lisa, a pre-service elementary school teacher:

1. Interviewer:	Would you please give an example of irrational number?
2. Lisa:	π
3. Interviewer:	Could you please give another example?
4. Lisa:	$\sqrt{2}$
5. Interviewer:	Could you please give another example?
6. Lisa:	[pause] Not from the top of my head.
7. Interviewer:	So, are there other irrational numbers, other than π and $\sqrt{2}$?
8. Lisa:	Maybe if we add π and $\sqrt{2}$ we will get another irrational number.
9. Interviewer:	I see, and what about $\sqrt{2} \times \pi$?
10. Lisa:	I'm not sure.
11. Interviewer:	And how about $\sqrt{17}$?
12. Lisa:	Maybe, not sure.

In what follows, we present only the lists of participants' responses. We note that these examples were provided following a repeated request for 'another' one and also for "something different". According to Watson and Mason (2005), request for another, and then another example, encourages people first to tinker with their first example and then to search in a different direction.

Responses of Bob, a pre-service elementary school teacher:

- 0.12112211122211112222 …
- 5.45445544455544445555544444 …
- Whatever, there are many of these …
- Maybe 2.124576435789… not repeating.

Responses of Tanya, a high school mathematics teacher:

- $\sqrt{2}$
- π
- $\sqrt{3}$
- $\sqrt{7}$
- square root of any prime number
- 0.12396764 … not repeating decimal
- any non-repeating decimal.

Responses of Paul, a mathematician:

- $100 + \sqrt{2}$
- $100 + \sqrt{17}$
- $200 - \sqrt{2}$
- $\sqrt{10,001}$
- $\sqrt[3]{1,000,001}$
- $300 - \sqrt[3]{1,000,001}$
- $80\sqrt{2}$

It is clear from the interview excerpt above that Lisa's personal example space of irrational numbers is limited to π and $\sqrt{2}$. She also suggests, hesitatingly, that the sum of these irrational numbers will give an irrational result. However, she is not sure about what the product is. Furthermore, she cannot decide whether $\sqrt{17}$ is an irrational number.

From the examples generated by Bob it is safe to conclude that his view of irrational numbers is guided by his prevalent positive disposition towards decimals. In fact, he is aware of two different types of decimal representations: while the mathematical requirement is for no repetition in digits, Bob specifies that lack of repetition is assured either by presenting a pattern which is a 'growing' pattern, rather than repeating, or by a 'random' sequence of digits.

It is interesting to note that the first two examples given by Tanya, π and $\sqrt{2}$, are identical to the examples of Lisa. However, Tanya is definitely able to extend these examples and suggest that the "square root of any prime number" and "any non-repeating decimal" exemplify an irrational number. These general rubrics demonstrate her conceptual clustering of the rich pool of examples.

Paul's task was harder, as the requested examples did not only have to be irrational, but also fall in a given interval. His method of creating these irrational numbers exemplifies that he starts with familiar irrationals and then adjusts them using arithmetic operations to fit the required interval. He also mentions cubic roots, but does not mention decimal representations or π. Even before the request for "something different", Paul has a rich variety in his examples as he attempts to step away from the 'obvious' rubrics (exemplified by Tanya). These observations of richness and generality are acknowledged in the framework we propose.

Multiple solution problems

Can you give an example of a problem that can be solved in different ways? This question was posed to a group of practising high-school mathematics teachers in an individual interview setting. We examine several of their responses. (Interview excerpts are translated from Hebrew by the authors.)

1. Interviewer: Can you give an example of a problem that can be solved in different ways?

2. Boris: No, nothing comes to my mind at this moment.

3. Interviewer: Think a little.

4. Boris: All that I can think about … We [in the class] talked a lot about … Let's say, [we need] to prove that any triangle is an isosceles triangle. Then it took a lot of time to think what was the mistake [in the proof]. And there were several ways to think about it … One student said: "I want to draw the figure precisely". This was something unconventional.

[5-10: Skipped – conversation about the student.]

11. Interviewer: Can you think about additional examples?

12. Boris: Well. In geometry you have a lot of examples. First of all we teach not to be locked on a particular tool. So that geometry and trigonometry will not be separated [fields]. [You can] look at a problem from different points of view … You can arrive at any result in different ways …

13. Interviewer: Can you think about different kinds of multiple solution problems?

14. Boris: Different kinds? Tools?

15. Interviewer: You said you solved problems in different ways. How can you distinguish between the different ways?

16. Boris: I still do not understand your question.

17. Interviewer: You talked about different solutions for geometry problems and about solving a problem with geometry and trigonometry [tools]. Can you elaborate on this a little?

18. Boris: Look, for example, at proofs of the reduced multiplication formula [direct translation from Hebrew – formulas like $(a + b)^2 = a^2 + 2ab + b^2$] in geometric ways. You can draw a square, divide it in parts and then combine the parts differently. Let's say this is an option. But you also can open the parentheses and add similar addends. This is what students know. So you can make the lesson more colorful.

Typically (of the teachers), Boris's immediate reaction (contribution 2) was "Nothing comes to my mind". Most of the teachers provided similar answers at the beginning of the interview:

Sarit: I don't know, I don't have an example now. But I remember that I did this a few times. I don't have an exact example.

However, as the interview developed, all the teachers recalled different mathematics topics having problems that could be solved in different ways. Most of the teachers (like Boris in contribution 12) mentioned geometry and connected geometry with trigonometry.

Ronit: This happens in the plane geometry. This week, for example, I solved a problem in a certain way. Since I teach trigonometry in this class as well occasionally, two lessons ago I taught the area of the triangle using trigonometry. I told them [the students], "I can show you how to solve the same problem using trigonometry". The same problem in plane geometry can be solved using trigonometry.

For geometry problems, most of the teachers like Boris (contribution 12: "In geometry you have a lot of examples") just mentioned the subject and did not provide an example of a specific mathematical problem that could be solved in different ways. An exception may be seen in the example that referred to Pythagoras's Theorem and several of its proofs. In other cases, similar to Boris's, instead of providing an example of a specific problem, the participants mentioned curriculum areas in which such examples can be found and the ways in which the tasks can be solved. For example, Boris (contribution 18) mentioned proving reduced multiplication formulas by algebraic manipulations and by using areas. However, he neither specified the formulas nor presented the proofs.

Similarly, several teachers mentioned the possibility of solving quadratic equations in different ways, *i.e.*, factoring a polynomial, completing the square or quadratic formula.

For example, Sohir talked about investigating a function, with and without the use of the second derivative:

Sohir: For example, in function investigation …, there are several methods. The second derivative helps to decide whether we have a minimal or maximal point, and accordingly we can decide when the function increases or decreases. Or we may use the first derivative and the snake [interval] method to find domains in which the function either increases or decreases.

We suggest that the examples (more precisely, topics) mentioned by the teachers in our interviews were situated in their classroom practice. The excerpts from the interviews with Boris (contrbution 12: "we teach not to be locked"; contribution 18: "this is what students know") and Ronit ("Since I teach trigonometry in this class") demonstrate that when triggered to think about mathematical examples of multiple solution problems the teachers move into pedagogical territory, search for examples in their classrooms and think about the ways of teaching. Not surprisingly, the teachers at junior-high school mentioned topics from their curriculum, and those at senior high school drew on the topics that they taught.

We expected that the teachers would provide a richer variety of multiple solution problems, however, the examples suggested by teachers were bounded by their teaching practices. We differentiate two types of examples according to their sources (Leikin and Levav-Waynberg [1])

- *craft-pedagogical*, examples of the first type related to teachers' pedagogical experience and the mathematics that they themselves learn when teaching – teachers often noted that examples of problems that can be solved in different ways were not constructed or chosen *a priori*, but emerged during mathematics lessons in which different solutions were suggested by the students

- *curricular*, examples of the second type, are associated with topics from the school mathematics curriculum that include different specific solutions for the same problem – among these are solving systems of linear equations (using substitution or linear combination); solving quadratic equations (by factorising, by quadratic formula, by completing the square); investigating functions (with or without second derivative (Sohir).

During our interviews, despite a request for an example of a *problem*, most teachers provided examples of curriculum *topics* rather than a specific problem that could be solved in different ways. Apparently, such problems were not part of the teachers' active repertoire of examples. The existence of such problems was acknowledged, but the teachers did not remember and could not readily reproduce specific examples during the interview. These observations of generality, richness and accessibility are acknowledged in our proposed framework.

Towards the framework

Our working assumption is that example generating tasks may serve as a research tool in studies that aim to describe and analyze participants' knowledge. We have provided some support for this assumption by examining several cases of example generation. However, in order to be implemented widely as a research tool, a framework for analyzing qualities and structures of example spaces of participant generated examples needs to be designed.

We recognize Watson and Mason's observation that "exemplification is individual and situational" (p. 50). 'Individual' being understood as dependent on the knowledge and experience of the learner and 'situational' meaning, framed by the prompt and circumstances in which such knowledge is presented. In order to make inferences about participants' knowledge from the examples they generate, we must control the situation, that is, the ways in which they are invited to provide examples. It is the underlying assumption of our framework that participants have ample opportunity to provide or construct examples, and that multiple examples are encouraged by asking for 'another and another' and for 'something different'.

The two kinds of example generation tasks – that of mathematical objects and of mathematical problems – may appear very different in their purpose and structure. The differences depend not only on the nature or complexity of the task itself, but also on the participants who respond to the task. Depending on the individuals, students or teachers, participant-generated examples may allow us to analyze mathematical or pedagogical knowledge. While students' examples reflect their mathematical knowledge, teachers' examples show both their mathematical and pedagogical knowledge. For practising teachers these two kinds of knowledge are intertwined, situated in their practice (Leikin and Levav-Waynberg [1]). Nevertheless, it is our view that similar characteristics of examples surface in responses to both tasks. These include, but are not limited to, the following components (previously signposted in the examples): correctness and accessibility, richness, and generality.

A description of an individual's knowledge or certain understandings is based, at times informally and implicitly, upon comparison with understanding possessed by an 'expert' – a mathematician or a teacher. As such, the suggested framework will provide tools for comparing personal (evoked) example spaces, as triggered by an example-generation assignment, with conventional example spaces, as generally understood by mathematicians. We suggest that when contextualizing individual activity in the community of practice, collective example spaces may shed light on the qualities of personal example spaces as connected to the professional or learning experiences of the participants.

Accessibility and correctness

Were the examples correct, that is, have they satisfied the conditions of the task? Were the examples generated with ease or with struggle? Were they 'pulled out of a sleeve', constructed using specific procedures, or selected from resources? Were there any procedures used for constructing examples or for checking that the conditions of the task were satisfied? Were the procedures used for example generation mathematically correct, elegant, or unnecessary?

For example, a procedure for generating a number that leaves a remainder of 1 in division by 2 by multiplying some number by 2 and adding 1 is mathematically correct but is absolutely unnecessary if understanding of parity is involved. Further, focusing on the excerpt with Cindy above, division with remainder is unnecessary to determine the remainder in division by 2 of an odd number. Carrying out this procedure shows that in Cindy's web of knowledge those two features of a number – oddness and remainder of 1 in division by 2 – are not connected.

Giving an example of a multiple solution task appeared to be rather complicated for the teachers and in most cases they could not formulate such a task promptly. The interviewer had to trigger teachers' reasoning with series of questions to help them in generating their examples. We did not observe clear procedures for constructing multiple solution tasks (*e.g.*, problem posing procedures). Additionally, in many cases, correctness of the examples could not be addressed due to their lack of concreteness. Since exemplification is situational, it can be the case that the teachers assumed that the interviewer was familiar with problems from the curricular topics they mentioned (Leikin and Levav-Waynberg [1]). However, we believe that in many cases lack of concreteness of the tasks provided by the teachers reflected the lack of their accessibility to the multiple solution tasks.

Richness

Did the examples vary in kind? Was there a fluency in any variety? Were examples routine or non-routine? How does the personal example space of a participant relate to conventional example space? How is the personal example space similar to/different from the collective example space? Were the examples situated in a particular context, such as curriculum or classroom experience?

Richness of an example space can be seen as an indicator of a particular concept image. In the case of irrational numbers, we see that Lisa's example space is rather 'poor', it includes only two elements, π and $\sqrt{2}$. In fact, other instances of example spaces limited to these two numbers were observed in prior research (Sirotic and Zazkis, 2007). For Tanya, π and $\sqrt{2}$ are also the first examples that come to mind, but her further examples point to a relative richness – they include additional square roots and also decimal representations. However, her examples, as well as the way in which she describes them, suggest that her knowledge is strongly situated in a school curriculum.

Paul's case is an illustration of how the richness of example spaces is triggered by the task. Without limits on the interval, his irrationals may not have been presented as sums or products. His image of irrationals is connected to roots and operations and does not rely, at least initially, on decimal expansion.

Richness of an example space may also hint at a learner's disposition. It is clear from Bob's examples that he focuses exclusively on the decimal representation of irrational numbers. Such a disposition towards decimals is limiting in decision making about irrationality of numbers and leads to potential errors (Zazkis and Sirotic, 2004). Another illustration of students' dispositions emerging from their examples is evident in the examples of arithmetic sequences described above. This disposition towards multiples appeared troublesome when participants were engaged in tasks – such as identifying whether a given element belongs to a given sequence – that concerned sequences of non-multiples (Zazkis and Liljedahl, 2002).

Teachers' responses to "give an example of a multiple solution task" provide a different perspective on the richness of the examples. When talking about the problem richness we address both the problem itself and its solutions. Similarly to Tanya's example space of irrational numbers analyzed above, both personal and the collective example spaces of multiple solution tasks were situated in the school curriculum. Moreover, the interviews revealed that teachers' knowledge is situated in their classroom practices. Mostly the tasks with solutions suggested by the teachers were routine. Examples provided by Boris – the teacher with the strongest mathematical background among the interviewees – demonstrate richness that is not typical for this group of participants. His examples include a non-routine problem of finding a mistake in an incorrect proof of a mathematical statement ("any triangle is isosceles") and a non-routine geometric approach to routine formulas of reduced multiplication. Thus, we suggest that richness of example spaces is strongly dependent on the participants' educational background. Whereas the majority of teachers' examples of multiple solution tasks are situated in the prescribed curricular and craft (practice) sources, non-routine tasks are usually borrowed from systematic sources of knowledge (for the distinction between craft, systematic and prescriptive teachers' knowledge, see Kennedy, 2002).

Generality

Were the examples specific or general?

Watson and Mason (2005) note that an example generation task "may bring to mind a single example, or a class of examples or a 'flavour' of possible examples" (p. 50). A 'flavour' is interpreted as an essence or a format that needs filling out.

Perception of generality is individual. While for one person $\sqrt{17}$ is a specific example, possibly a variation on the 'classical' example of $\sqrt{2}$, for another it may serve as a representative of a class of 'roots of primes' or 'square roots of numbers that are not perfect squares', even if the class is not described explicitly.

In the usual context of mathematics education, 'general' appears to be more valuable than 'specific'. Usually, we strive for "seeing the general in the particular" (Mason and Pimm, 1984) and appreciate generalized or generalizable solutions. However, a word of caution. While some general examples, characterizing classes of objects, may be seen as an indication of mathematical understanding, other general examples may point to deficiencies in understanding. In other words, the 'generality' may serve as a generator, but may also serve as a protective shield. For example, the claim that a square root of any prime exemplifies an irrational number is a format – or, in Watson and Mason's terms, a 'flavour' – for generating (an infinite number of) examples. However, Cindy's claim above, that a number that leaves a remainder of 1 in division by 2 is an odd number does not serve for her as a generator of examples. Without further probing, this gives an illusion of understanding, when, in fact, there is an inability to come up with a specific example. As such, we consider not only the "flavour of possible examples", but also individual's ability to work with this flavour.

Paraphrasing Mason and Pimm (1984), notwithstanding the general, we are interested in participants' ability of "filling the general with the particular". Additional illustration for the unfilled 'example flavour' is in the lack of concreteness in the examples of the multiple solution tasks provided by teachers. This is in contrast to the anticipated example that could include a precise formulation of a problem and its different solutions.

Conclusion

We adopt the view that "to understand mathematics means, among other things, to be familiar with conventional example spaces" (Watson and Mason, 2005, p. 64). From this position, we believe that learners' example spaces, and their relationship to the conventional ones, provide a window into their understanding of mathematics. We considered two kinds of example spaces generated by participants: examples of mathematical concepts and examples of mathematical tasks. By analyzing revealing features of these examples we suggested a possible lens through which learners' example spaces can be viewed in order to examine the learners' knowledge and understanding. The suggested framework is a guideline, it is neither comprehensive nor complete; some of the components are not applicable for some example generation tasks and example spaces while some possible characteristics of example spaces are not featured in the framework. Nevertheless, we consider it to be a compelling starting point for further development and successive refinement.

Notes

[1] Leikin, R. and Levav-Waynberg, A. (in press) 'Exploring mathematics teacher knowledge to explain the gap between theory-based recommendations and school practice in the use of connecting tasks', *Educational Studies in Mathematics*.

References

Ginsburg, H. (1997) *Entering the child's mind: the clinical interview in psychological research and practice*, Cambridge, UK, Cambridge University Press.

Kennedy, M. (2002) 'Knowledge and teaching', *Teacher and Teaching: theory and practice* **8**, 355-370.

Leinhardt, G. (1993) 'On teaching', in Glaser, R. (ed.), *Advances in instructional psychology*, 4, Hillsdale, NJ, Erlbaum, pp. 1-54.

Mason, J. and Pimm, D. (1984) 'Generic examples: seeing the general in the particular', *Educational Studies in Mathematics* **15**, 277-289.

Sirotic, N. and Zazkis, R. (2007) 'Irrational numbers: the gap between formal and intuitive knowledge', *Educational Studies in Mathematics* **65**(1), 49-76.

Skemp, R. (1987, second edition) *The psychology of learning mathematics*, London, UK, Penguin Books.

Watson, A. and Mason, J. (2005) *Mathematics as a constructive activity: learners generating examples*, Mahwah, NJ, Lawrence Erlbaum.

Zazkis, R. and Liljedahl, P. (2002) 'Generalization of patterns: the tension between algebraic thinking and algebraic notation', *Educational Studies in Mathematics* **49**, 379-402.

Zazkis, R. and Sirotic, N. (2004) 'Making sense of irrational numbers: focusing on representation', *Proceedings of the twenty-eighth annual conference of the International Group for the Psychology of Mathematics Education*, **4**, Bergen, Norway, Bergen University College, pp. 497-505.

Zhu, S. and Simon, H. (1987) 'Learning mathematics from examples and by doing', *Cognition and Instruction* **4**, 137-166.

[These references follow on from page 39 of the article "Mapping mathematical communities: classrooms, research communities and masterclass hybrids" that starts on page 34 (ed.)]

Doyle, W. (1986) 'Classroom organisation and management', in Wittrock, M. (ed.), *Handbook of research on teaching*, New York, NY, Macmillan.

Gergen, K. (1999) *An invitation to social construction*, London, UK, Sage Publications.

Hardman, F., Smith, F., Mroz, M and Wall, K. (2003) 'Interactive whole class teaching in the National Literacy and Numeracy Strategies', The British Educational Research Association Annual Conference (BERA 2003), Heriot-Watt University, Edinburgh, available online at http://www.leeds.ac.uk/educol/documents/00003267.htm, accessed on 28th April, 2007.

Hersh, R. (1998), *What is mathematics, really?*, London, UK, Vintage.

Julie, C. (2002) 'The activity system of school-teaching mathematics and mathematical modelling', *For the Learning of Mathematics* **22**(3), 29-37.

Kelly, P. (2006) 'What is teacher learning? A socio-cultural perspective', *Oxford Review of Education* **32**(4), 505-519.

Lave, J. (1997) 'The culture of acquisition and the practice of understanding', in Kirshner, D. and Whitson, J. (eds), *Situated cognition: social, semiotic and psychological perspectives*, Mahwah, NJ, Lawrence Erlbaum.

Lave, J. and Wenger, E. (1991) *Situated learning: legitimate peripheral participation*, Cambridge, UK, Cambridge University Press.

Mercer, N. (1995) *The guided construction of knowledge*, Clevedon, UK, Multilingual Matters Ltd.

Pimm, D. (1987) *Speaking mathematically. Communication in mathematics classrooms*, London, UK, Routledge and Kegan Paul Ltd.

Pratt, N. (2006a) ''Interactive' teaching in numeracy lessons: what do children have to say?', *Cambridge Journal of Education* **36**(2), 221-236.

Pratt, N. (2006b) *Interactive maths teaching in the primary school*, London, UK, Paul Chapman Publishing.

Schoenfeld, A. (1985) *Mathematical problem solving*, New York, NY, Academic Press.

Schoenfeld, A. (1996) 'In fostering communities of inquiry, must it matter that the teacher knows 'the answer'?', *For the Learning of Mathematics* **16**(3), 11-16.

Solomon, Y. (1998) 'Teaching mathematics: ritual, principle and practice', *Journal of Philosophy of Education 32*(3), 377-390.

Stewart, I. (1997) *The magical maze: seeing the world through mathematical eyes*, London, UK, Phoenix.

van Oers, B. (2001) 'Educational forms of initiation in mathematical culture', *Educational Studies in Mathematics* **46**, 59-85.

Wedege, T. (2002) 'Numeracy as a basic qualification in semi-skilled jobs', *For the Learning of Mathematics* **22**(3), 23-28.

Wenger, E. (1998) *Communities of practice: learning, meaning and identity*, Cambridge, UK, Cambridge University Press.